Local and Global Geometric Methods for Analysis Interrogation, Reconstruction, Modification and Design of Shape

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Abstract

This paper gives an overview of some recent methods useful for local and global shape analysis and for the design of solids. These methods include as new tools for global and local shape analysis the Spectra of the Laplace and the Laplace-Beltrami Operator and the Concept of stable Umbilical Points i.e. stable singularities of the principal curvature line wire frame model of the solid's boundary surface. Most material in this paper deals with the Medial Axis Transform as a tool for shape interrogation, reconstruction, modification and design. We show that it appears to be possible to construct an intuitive user interface that allows to mould shape employing the Medial Axis Transform. We also explain that the Medial Axis and Voronoi Diagrams can be defined and computed as well on free form surfaces in a setting where the geodesic distance between two points p, q on a surface S is defined by the shortest surface path on S joining the two points p, q. This leads to the natural and computable generalized concepts of geodesic Medial Axis and geodesic Voronoi Diagram on free form surfaces. Both can be computed with a reasonable speed and with a high accuracy (of about 12 digits when double floating point arithmetic is used for the computations).

1 Introduction

As the information age is rapidly changing our modern world, the relative contributions of different components defining our economies are changing rapidly as well. A major part of economic value creation in our modern society is not given any more by fabricating single physical objects, but is created within the context of information exchange. An essential initial part of the effort needed to fabricate physical objects is in the design process creating the 3D digital model description of the object.

In the past these 3D solid model descriptions were presented with a fairly restrictive shape variety, by 2D - drawings (blueprints). Today these 3D solid objects are typically described with CAD systems using digital data sets. Here the richest shape variety can be reached by solids with free form boundary surfaces that are typically defined by Non-Uniform Rational B-Spline (NURBS) surface patches. Consequently the most important and most fundamental part of the value creation process for 3D object consists in creating the digital 3D solid model bounded by free form surfaces.

2 Two Fundamental Problems in Solid Modeling

2.1 Approximate Shape Identity

There exist several basic problems in the context we address here. The first problem is to answer the question: Are two solid objects that are presented to a computerized system in (perhaps completely) different representations approximately describing a solid object with the same shape? Here one of the objects might be described with patches of splines or as a solid with a facetted boundary surface or even by using implicit functions for parts of its boundary surface! One of the two solids might have been scaled. In the end we want to decide if the solids agree approximately after appropriately scaling or moving one of them in space. This check for approximate shape identity is then of course also

[1] This was paper written while F.-E. Wolter was a visiting professor at MIT during his research sabbatical in the winter term 1999/2000.

closely related to the problem to facilitate appropriately the Data Exchange beween different systems describing solids in different ways as indicated above employing e.g. implicit functions, NURBS or facetted boundary surfaces.

Therefore it may be necessary to find an appropriate practical standard as to exchange the data describing a solid beween different CAD systems that may use completely different definition standards as to describe the shape, cf. the options already indicated above. For this we would have to create approximate representations allowing convenient conversion methods! This question of approximate conversion is then immidiately related to the problem to find approximations using a small number of data and satisfying still certain accuracy requirements. The latter problem can be viewed to belong to the area of data compression!

2.2 Intuitive User Interface to Mould a Rich Shape Variety of Solids

The other fundamental problem related to the solid in space is the development of appropriate methods to design shape in a way that can be perceived as human user friendly allowing considerable shape varieties and offering also the possibility to appeal to human intuition when moulding shape. Hence the system should have a nice intuitive user interface that relates to the human haptic interaction with shape when moulding and perceiving shape. The latter feature is desirable because e.g. shape designers for the car industry still refuse to use current CAD tools to design shape as the CAD systems (at least according to the requirements of the designers) do not properly allow a natural haptic perception of the designed shape. Hence design practioners (being usually sensitive artists) until today prefer clay moulding to design shape because the latter offers an intuitive and natural haptic feedback and control of the design process. *We as human beeings have to accept that our haptic (tactile) analog interaction with the world is in some sense perhaps our most direct and fundamental interaction with the world. The human skin is a huge sense organ perceiving temperature, pressure, vibrations, roughness, recognizing palpable symmetries, recognizing 2D and 3D - (sculptured) shape structures and most importantly perceiving (and giving) via human touching in the most fundamental ways affection, love and consolation.* It is generally accepted that without perceiving and giving the latters our human existence would be miserable and empty.

3 History of this Paper's Material

This paper will discuss and will give an overview (generally without proofs) of concepts and results that address both fundamental questions presented above in 2.1 and 2.2. However the main emphasis will be on the second subject discussing background and suggesting possibilities of designing shape and suggesting an interface offering intuitive design *possibilities resembling shape moulding*. The theoretical considerations behind this paper reflect some important directions of the lead author's own research during the past twenty years. As far as the Medial Axis is involved this paper is essentially based on F.-E. Wolter's theoretical graduate research originating in his thesis work on the cut locus, see [6] or even earlier see [9]. In some meta sense his by now fairly old thesis [6] covers (the difficult parts of) most material presented in [2], the latter paper is perhaps mainly restating the special Euclidean Cases of more general results contained in [6]. Nonetheless despites its theoretical relevance for this field without the development of appropriate programs and visualization systems all these theoretical considerations in [6], [2] would have remained (may be interesting, beautiful) but only theoretical fantasies. However this changed with the lead author's arrival at the University of Hannover when he started teaching graphics and geometric modeling courses in summer 1995. Since then a considerable number of bright and dedicated students have been working with him and helped to put some of his old (theoretical) visions into concrete prototype programming systems proving that the theoretical considerations would indeed finally yield reasonable perhaps even efficient algorithms. Therefore the material sketchily covered in this paper reflects also the efforts of many of F.-E. Wolter's students during the past four years. Those students helped to develop the research mostly through contributions in their master theses and sometimes in their senior theses. The students who did work in the context of this paper were O. Sniehotta with his senior thesis and R. Kunze, M. Baer with their master theses work that finally lead to the computation of geodesic medial curves, geodesic Voronoi Diagrams and the geodesic Medial Axis on bordered surfaces. The latter problems posed computational difficulties that had been resistant against computational efforts for quite a while. Most intensely involved in particular in the efforts related to the geodesic Medial Axis was the lead author's former Ph.D. student T. Rausch whose thesis work (among other subjects) was dealing heavily with the analytical, geometrical and numerical aspects of precise and reasonably fast differential equation methods useful to compute the geodesic Medial Axis (see [3] and [4]). In the Medial Axis research in the Euclidean Case there were involved O. Etzmuss (with his senior thesis for the 2D-Curved-Boundary case), P. von Grumbkow for 2D- and 3D solids with piece wise linear boundary and A. Howind who in his master thesis discussed the subject of modifying (and designing) a solid's shape by modifying the associated maximal disc radius function defined on the original Medial Axis. This research idea had first been submitted by the lead author within a fairly large group proposal to the German DFG in October 1996. Unfortunately this proposal did not get funded. Consequently some of the planned research was then done not as sponsored research but within some master theses projects. As a result of this P. v. Grumbkow's and A. Howind's master theses show that considerable parts of the proposed research ideas would indeed work out and this paper here now uses many of A. Howind's figures to illustrate this. Furthermore related to the Medial Axis is also the master thesis of the lead author's former student A. Kaiser discussing computations of orthogonal projection curves. A. Kaiser studied delicate situations that arise when computing on a surface S a surface point nearest to a given space point p when p is approaching a curvature center of the surface S.

The precedingly sketched material was mainly related to the Medial Axis in the Euclidean and in the Geodesic Case. It will turn out later that this Medial Axis material will already cover all of the items mentioned in the title of this paper. However we shall also briefly sketch two additional methods that are "Local and Global Geometric Methods for Analysis, Interrogation of Shape" as it was announced in this papers title. These two methods can then be viewed as methods supporting the treatment of the fundamental problem 2.1 described above).

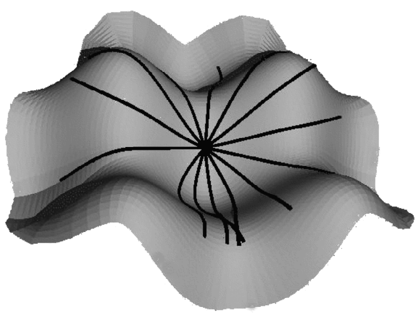

Figure 1. Geodesics on a wave like parametric surface defined by trigonometric functions

The first of those additional methods is related to more recent ongoing research using also work contained in two master theses investigating how the Spectrum of the Laplace Operator or how the Spectrum of the Laplace-Beltrami Operator could well be used to analyse shape similarities. It is theoretically known that those Spectra (being Isometry invariants of the domains or surfaces) must agree for two surfaces if the two surfaces are congruent or even only isometric, cf. [8]. Hence this gives us here a necessary criterium that must be fulfilled if two surfaces are tested for global isometry. For this research subject (as preliminary investigations) two master theses of F.-E. Wolter's students T. Howind and T. Altschaffel are now finished. T. Howind treats in his master thesis planar domains, while T. Altschaffel has been investigating the spectrum of the Laplace-Beltrami Operator for some classes of surfaces in 3D.

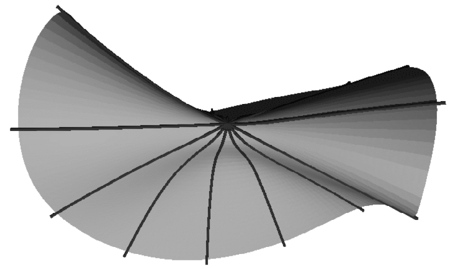

Figure 2. Geodesics emanating from a common point on an implicitly defined cubic surface

The second of these additional methods testing shape similarity has been described and discussed with many details in a paper that the first author wrote with his former (and current) MIT colleagues T. Maekawa and N. Patrikalakis, cf. [5]. That paper considers stable umbilics as a fairly robust surface shape feature characterizing (stable) singularities in a net of principal curvature lines on the surface. If a surface patch is only mildly deformed then these stable umbilics occur again on the deformed surface. Hence those stable umbilics provide local or semi local shape features useful to detect (or also to disprove) local similarity between surface regions on different surfaces. One could disprove local shape similarity of surface patches if their umbilical point structures disagree!

The preceding retrospective outline describing the genesis of this paper's material within the lead author's academic biography is not quite complete. During the lead author's tenure at Purdue University 1987-1989, he pursued as sole principal investigator a research project supported by the Army Research Office (Grant DAAL-03-88-K0186). Out of that project called **"Project Riemann"** came an (autonomous) programming system with the following capabilities: A user could input into the system expressions defined by elementary functions describing surfaces either implicitely or parametrically. Then the system would perform first symbolic and finally numerical calculations that would compute surface geodesics (starting from a given surface point), cf. figures 1-3. The system could also compute curvature line wire frame models of surface parts (see the dark grey lines in figure 3). The system would also visualize the computed geodesics and curvature lines. In summer 1988 under the lead author's guidance and supervision the programming of the project Riemann system was done by a group of enthusiastic and talented mostly undergraduate (sophmore) students at Purdue University. That group included B. Johnson, J. Lambers, S. Cutchin, S. Goehring, and T. Hausmann the latter as a masterstudent. Figure 1 shows a result computed by the Project Riemann system displaying geodesics emanating on a wave like surface from a center point. *It appears that in 1988 -(in those now somewhat early days of computational surface geometry)- the Project Riemann System was the first program system providing all the aforementioned capabilities.* Figures 1-3 have been prepared by the first author's student A. Kaiser at the Welfen Laboratory using the "Project Riemann" code.

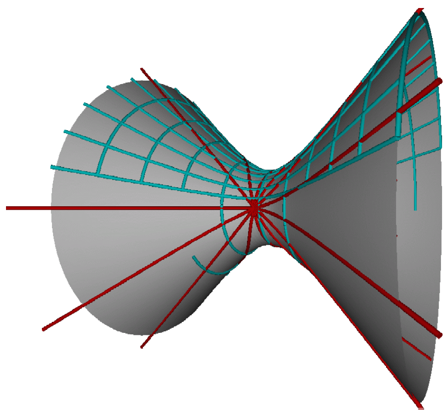

Figure 3. Hyperboloid

After this retrospective the lead author wants to say thank you to all the students he worked together with on research projects during his academic career. The names listed above present only a part of those students. It here only acknowledges the students as far as they were involved in work related to the general subject of the current paper. It may be noticed as interesting that the vast majority of students involved in the related research projects were either undergraduate or master students. Hence this research resulted from activities that could be understood as part(s) of advanced undergraduate education. *This is no coincidence because it reflects the lead author's belief that ultimately the best, most intense and really lasting learning is done while doing research i.e. while inventing, seeing, understanding, discovering something original. The reason for this may be the universal very human desire to be and to do something unique and lasting and gain (unique) identity with a lasting impact beyond our physical existence from this. At least in some ways everybody appears to have some artistic desires and some exploring mind.* Therefore according to the lead author's experience whenever students sensed that they were exploring completely new roads and ideas and solving new challenging tasks that had resisted against research efforts then the students could develop tremendous energy, curiosity, enthusiasm and patience that they could not develop easily when working on routine problems.

4 Two Methods for Shape Interrogation and Classification

We start with a short description of a global method to test similarity of surface shapes employing Eigenvalues of the Laplace Operator and the Laplace-Beltrami Operator. Hence this will suggest an approach to treat the fundamental problem 2.1.

4.1 The Laplace-Beltrami Operator for Surfaces

In order to avoid more lengthy descriptions we use simplifications that are not always completely precise in order to explain the relevant concepts. Let S be a surface patch presented by a twice continuously differentiable parametric map $r(u, v)$ defined on a simply connected planar domain D with smooth boundary curve ∂D. More general we may require that D fullfills the condition that every point in D has an open neighborhood U in D such that U can be mapped with an infinitely often continuously differentiable diffeomorphism onto a convex set in the plane; (an n-times continuously differentiable diffeomorphism must have an inverse mapping being n - times continuously differentiable as well).

In order to define the Laplace-Beltrami Operator we need to introduce some notations from differential geometry. We need the first fundamental tensor matrix whose elements E, F, G are defined by the inner products of the partial derivatives of the surface parametrization $r(u, v)$. Introducing now the subsequent notations will be convenient. For this we also assume below that

$$\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$$

is an infinitely often continuously differentiable function.

$$(g_{ij})_{l \le i,j \le 2} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$(g^{ij})_{l \le i,j \le 2} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

$$u^1 := u, u^2 := v_i, \varphi_i := \frac{\partial \varphi}{\partial u_i}$$

Furthermore the coefficient functions

$$\Gamma_{ij}^k$$

are called Christoffel Symbols, they can be expressed in terms of the elements of the first fundamental tensor matrix and its derivatives only, see [7].

Now we can define the Laplace-Beltrami Operator as a differential operator assigning to a function

$$\varphi : \mathbb{R}^2 \to \mathbb{R}$$

the subsequently defined function:

$$\triangle\varphi := \sum_{i,j=1}^{2} (\frac{\partial}{\partial u^j}[g^{ij}\varphi_i] + \sum_{k=1}^{2})\Gamma^k_{jk} g^{ij}\varphi_i)$$

By this we obtain a linear Operator defined say on the vector space of infinitely often differentiable real valued functions that are defined to be zero on the boundary of the domain D. It is well known that the Eigenvalues of the Laplace-Beltrami Operator are all not negative and yield the same values for two surfaces A, B that are isometric. Isometric means there is a diffeomorphism from A to B that maps every smooth curve on surface A onto an equally long curve on B. This "Laplace-Beltrami Operator (spectrum) concept" works for closed surfaces as well! Clearly if two solids are congruent then their boundary surfaces must be isometric and hence their Spectra (collections of Eigenvalues) of their respective Laplace-Beltrami Operators must agree. For planar surface patches (being isometric to planar domains) the general Laplace-Beltrami Operator becomes the classical Laplace Operator. Investigating closed surfaces it would be desirable to compute the Spectrum of the 3-dimensional Laplace Operator with respect to the solid body in the 3-dimensional Euclidean space.

Looking at a variety of sample surfaces our preliminary investigations contained in the master thesis of A. Howind and T. Altschaffel (dealing especially with the Laplace-Beltrami Operator), indicate that those spectra appear to show promise to be useful for the construction of new shape invariants helpful to analyse shape similarities. For doing the actual computation of the Eigenvalues one would usually work with a variational form of the given Eigenvalue problem stated above in its partial differential equation version. Hence we have treated the given Eigenvalue problem in its equivalent variational form (evaluating integrals) employing Finite Element methods.

4.2 Umbilical Points and Curvature Line Wire Frame

We consider solids with a regular curvature continuous boundary surface i.e. the boundary surface can be locally represented by surface patches having (at least) twice continuously differentiable parametrizations whose Jacobians always have rank two. Any boundary surface of the solid can be represented by a wire frame model built by (orthogonal) families of principal curvature lines, shown in figure 4 (see also the grey lines in figure 3). This wire frame model is intrinsic to the surface (point set). This means it is independent of the chosen (local) parametrization of the surface that might be used to compute the curvature line wire frame.

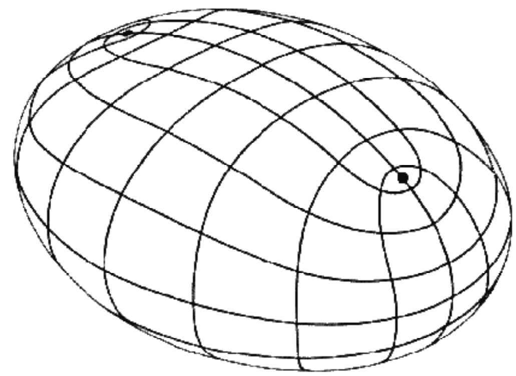

Figure 4. Curvature lines and two umbillics indicated by dots on an ellipsoid

Generically this curvature line wire frame surface model is built by rectangular meshes. However there do typically exist also some exceptional points on the surface where the otherwise rectangular wire frame structure is violated i.e. here the wire frame has singularites (see figure 4, the two strong dots). These special singular points are umbilics of the given surface i.e. at an umbilical point p all normal curvatures agree for all tangent directions at the surface point p. (If a closed surface S is homeomorphic to a sphere then it follows from a topological argument that S must have at least one umbilic.) One could also say that the minimal and maximal principal curvature at an umbilic agree or equivalently every tangent direction at an umbilic is a principal curvature direction. A principal curvature direction at a point p can be defined as a tangent (angular) direction β (at the given point p) where the normal curvature assumes a local extremum when compared with normal curvatures at p that correspond to angular directions close to β. The curvature line pattern near an umbilic has been classified into different types. There only exist three different stable types of umbilics where the curvature line pattern will remain stable (unchanged) with respect to small deformations of the surface!

Those three stable umbilics are **star, monstar, lemon**, see eg. Maekawa, Wolter, Patrikalakis 1996 in [5]. The three characteristic curvature line pattern in a neighborhood of the respective stable umbilic are described in the three subsequent figures, adapted from [5]:

Figure 5: Left image a star, right a monstar umbilic, both have three curvature lines passing through the umbilic. The three tangents for the three curvature lines passing through the umbilic in the monstar case are contained in a right angle. They are not contained in a right angle in the star case.

Figure 6 shows an umbilic of lemon type. There is only

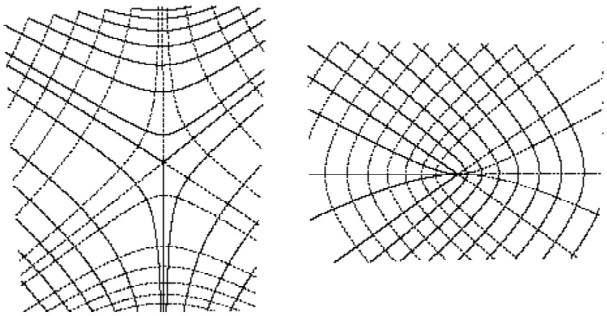

Figure 5. Star and Monstar Umbilic

one curvature line passing through this umbilic.

We now observe the following characteristic properties: Three lines of curvature are passing through umbilics of star and monstar type, cf. figure 5, while only one curvature line is passing through an umbilic of lemon type, cf. figure 6. The criterion distinguishing the star and monstar is that for the monstar case the three tangent lines of the curvature lines passing through the umbilic are contained in a right angle while in the star case they are not, cf. figure 5.

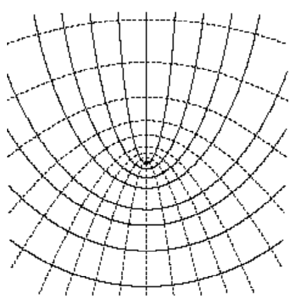

Figure 6. Lemon Umbilic

If two curvature continuously closed surfaces S and S' are obtained from each other by deformations (being sufficiently small) then the stable umbilics on both surfaces can be matched in one to one correspondence! Furthermore in the generic case i. e. if there are only stable umbilics on both surfaces S and S' then it is possible to find appropriate curvature line wire frame models for both surfaces S and S' that can be aligned and matched quite well. Hence we here have concepts and methods that are quite appropriate to support "tests for local and global approximate shape identity of solid objects" contributing to treat the fundamental problem 2.1.

4.3 Medial Axis Transform as Basis for Shape Interrogation and Intuitive Design

In this paper we suggest concepts for an interface with an intuitive design control resembling shape moulding employing the Medial Axis Transform. Therefore we have to explain the Medial Axis Transform first and we also want to discuss and show that the Medial Axis Transform is a natural tool to characterize shape features of a solid body in space. The importance of the Medial Axis may have been conjectured first by the late H. Blum whose paper [1] tries to explain in heuristical ways that the Medial Axis should be of interest in the context of biological shape.

Definition *(Medial Axis and Medial Axis Transform): The* **Medial Axis** *of a solid D in the n-dimensional Euclidean space is a subset of D containing all points being center of a disc of maximal size that fits in the solid D. We associate with the Medial Axis of a domain D the* **maximal disc radius function**, *assigning to a Medial Axis point p the radius of the maximal disc with center p. We assume in the Medial Axis case that D is closed and that the boundary ∂D of D is a topological (not necessarily connected) hypersurface of the Euclidean Space. The Medial Axis together with the associated maximal disc radius function constitute the* **Medial Axis Transform** *of a solid*

An example of the Medial Axis of a domain D is shown in figure 7 from the senior thesis of the first author's former student O. Etzmuss.

To simplify the matters we shall generally assume (unless we say otherwise) that the boundary of the solid is curvature continuous i.e. that it can be parametrized locally by a regular twice continuously differentiable function whose first derivative has maximal rank. (There are many possibilities of relaxing these curvature continuous smoothness assumptions for the boundary, see [2].) We consider here in particular two classes of solids i.e. those where the solid is 2- or 3-dimensional, hence the solid's boundary may be built by arcs or surface patches.

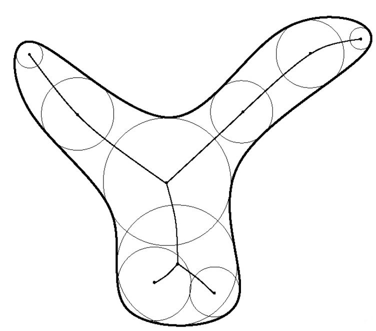

Figure 7. Medial Axis

Figure 7 shows for a 2D-solid the Medial Axis whose boundary points are the four start vertices of medial arcs. The Medial Axis in this figure is built by five medial arcs of which four have boundary vertices.

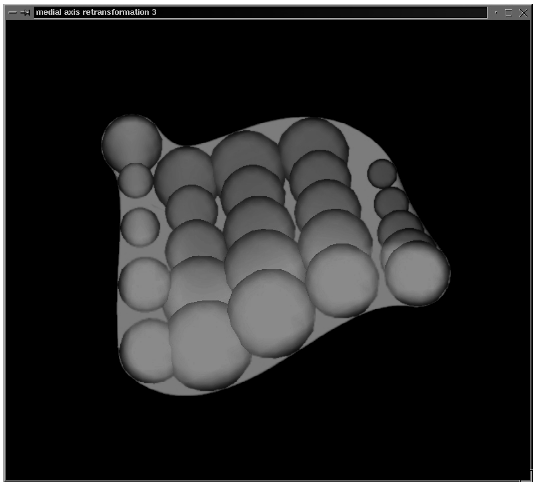

Figure 8. Medial Modeler

Figures 8, 9 and 11 are adapted from the first author's student A. Howind's master thesis. Figure 8 shows a collection of maximal spheres contained in the solid. Figure 9 and later figure 11 depict the boundary curve of the Medial Axis of the 3D-solid whose boundary surface is partly shown in figure 9 and in figure 11 by parts of spherical caps. The Medial Axis of this solid here only contains one medial surface patch. All the points of the medial surface are centers of maximal spheres that still fit into the solid, cf. figure 8.

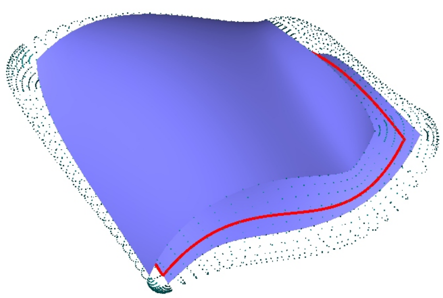

Figure 9. Medial Modeler

Assuming curvature continuous boundary of the solid, the Medial Axis will not have points that accumulate densely in some open subset of the ambient Euclidean space, see [2]. Under practically reasonable assumptions that the surface boundary is say defined by splines (or more generally by piecewise polynomial or real analytic or even piece wise analytic functions) it can be shown that the Medial Axis is a semianalytic set that can be defined using zero sets and subzero sets of real analytic functions. The Medial Axis can here be locally assembled by parts contained in finitely many zero sets of analytic functions. Those functions (often) arise from the condition that (generically) the Medial Axis can be assembled locally by a finite number of **medial sets** containing points being **equidistantial to two** appropriately chosen (disjoint) **boundary arcs or surface patches** contained in the solid's boundary. This implies roughly spoken that under practical assumptions the Medial Axis is (generically) built by a finite union of arcs or (usually bordered) medial surface pieces that can be computed numerically if they are not too many.

In order to state the preceding and the subsequent remarks on the analytical structure (zero-set structure) of the Medial Axis mathematically precisely and completely one would have to present a longer discussion containing more formal elements. We have chosen a simplified (not quite complete and correct) description assuming still real analytic boundary for the solid.

Computations of the Medial Axis are also possible if the boundary is not curvature continuous but e.g. given by planar facets. Figure 10 illustrates this situation.

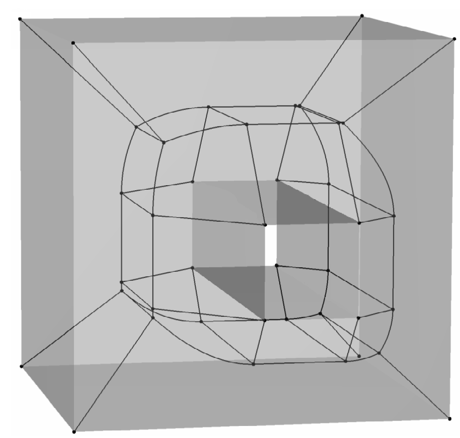

Figure 10. Medial Axis Structure indicated for a three dimensional Torus

Figure 10 is adapted from the master thesis of the lead author's student P. v. Grumbkow. This thesis deals with developing an algorithm and a program that could compute the Medial Axis of a solid object bounded by planar facets. Figure 10 illustrates the Medial Axis of a 3D-torus being a solid with a (topological) torus boundary surface. This solid is created by removing an open rectangular solid from a solid cube. The Medial axis is generally defined as the set of centers of maximal discs that fit in the solid. The Medial Axis of 3D-Solids is typically built by a union of smooth bordered medial surface pieces with smooth boundary curves. Figure 10 depicts the boundary edges and boundary vertices of some smooth medial surface pieces (the union of which yields the Medial Axis).

Now we return to the case where the solid's boundary is curvature continuous. Here the boundary points of the Medial Axis i.e. points that are contained in the boundary of only one medial surface piece (3D-solid case) or only one medial arc (2D-solid case) are given by points being curvature centers with respect to appropriate boundary points. Here in the 2D-solid case any Medial Axis boundary point is a curvature center of the solids boundary curve at a point where the boundary curvature attains a local maximum. In the 3D-solid case boundary points of the Medial Axis are given by curvature centers with respect to the minimal principal curvature radius of the boundary surface, the latter curvature radius function evaluated at the appropriate boundary points. Let us assume that in the 3D-solid case some boundary points on a medial surface patch are given by a curve b (cf. figure 11). Then typically the (minimal) curvature radius function has a local minimum when this radius function is restricted to *any single surface curve that crosses the projection curve* $P(b)$ *orthogonally*, cf. figure 11; here the projection curve $P(b)$ is obtained as an orthogonal projection of b onto the boundary surface i.e. every point of $P(b)$ is joined via the surface normal to its corresponding (minimal radius curvature center) point on b. It is not difficult to prove, that the curve $P(b)$ is a minimum principal curvature line on the solid's boundary surface. The aforementioned family of orthogonal curves has tangential directions at points on $P(b)$ being maximum principal curvature directions of the solid's boundary surface.

As already indicated, the preceding statements can be understood more easily by looking at the illustrating figures 7 and 11. All the boundary vertices of the Medial Axis in figure 7 correspond to local curvature maxima of the boundary curve. Figure 11 shows some spherical caps beeing parts of the solid's boundary surface. It also shows the boundary curves of the Medial Axis of the solid depicted by the closed curve in the figure. Projecting the Medial Axis boundary edge b on to the boundary surface yields a maximum principal curvature line $P(b)$.

The following result shows why the Medial Axis is in a topological sense fundamental for the shape of a solid as it essentially contains the Homotopy type of a solid because it is a deformation retract of the solid. We have the following result, from [2]:

Topological Shape Theorem of the Medial Axis: *The Medial Axis $M(D)$ contains the essence of the topological type of a solid D.*

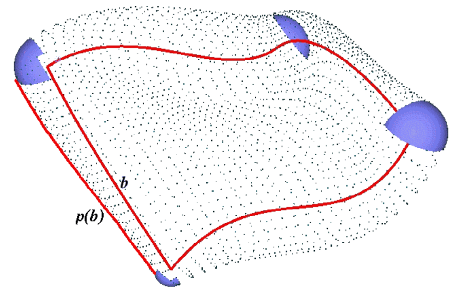

Figure 11. Medial Modeler

Let ∂D be C^2- smooth (or let ∂D be 1-dimensonal and piecewise C^2- smooth, with $D \subset R^2$) Then the Medial Axis $M(D)$ is a deformation retract of D thus $M(D)$ has Homotopy type of D.

The proof of this theorem shows that it is possible to define a Homotopy $H(x,t)$ as explained below the next figure describing a continuous deformation process of the solid D. This deformation process is depending on the time parameter t. The deformation starts with the solid being in the figure a rectangle with a circular hole. During the deformation points are moved along the shortest segments starting at the solids boundary ∂D until the segments meet the dotted Medial Axis. The shortest segments are indicated by arrows in figure 12.

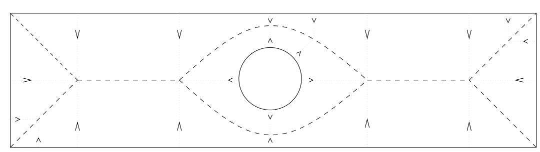

Figure 12. Deformation retract

Homotopy

$$H(x,t) : (D \backslash \partial D) \times [0,1] \to (D \backslash \partial D)$$

such that

$$\begin{aligned} H(x,0) &= x \; \forall x \in D \backslash \partial D \\ H(x,t) &= x \; \forall x \in M(D) \\ H(x,1) &= R(x) \text{ with } R : D \backslash \partial D \to M(D) \backslash \partial D \end{aligned}$$

Definition of the Homotopy:

$$H(x,t) : x + t d(x, \psi(x)) \nabla d(\partial D, x)$$

$d(x,y)$ denotes the function describing the distance between variable points x, y; $\nabla d(x,y)$ describes the gradient of the distance function $d(x,y)$. $\psi(x)$ is defined as point where the extension of a minimal join from ∂D to $x \in (D \backslash \partial D)$ meets $M(D)$.

5 Geodesic Medial Axis and Geodesic Voronoi Diagrams

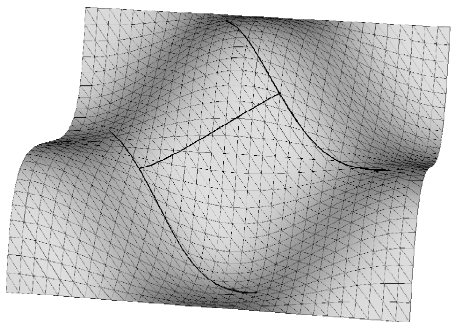

Figure 13. Shortest geodesic joining two surface arcs

So far we have been discussing the Medial Axis in the Euclidean Case. In order to define the Medial Axis we need to be able to compute a distance disc $D(p,r)$ with center p and radius r. $D(p,r)$ contains all points x that have a distance $d(p,x)$ not larger than r to the point p. During the last years we could show that it is possible to also compute the Geodesic Medial Axis and Geodesic Voronoi Diagrams on a free form surface S, see [3], [4]. They are defined like their Euclidean counter parts. However now the distance $d(p,x)$ between two surface points p, x is given by a shortest surface path on S joining the points p and x. This shortest path gives the (minimal) geodesic (surface) distance between p and x. The path may not be a Euclidean Segment because the latter may not be contained in the surface S. Figure 13 is illustrating a related example. This figure shows a surface curve whose length yields the minimal geodesic distance between any two points contained in the (different) sets joined by the curve. It is adapted from the doctoral thesis of the lead author's student T. Rausch. This figure shows two curves (point sets) on a surface that are joined by a minimal possible surface path. The length of this path gives the shortest distance on the surface between any two points in the two sets that are joined in figure 13.

Figure 14 is adapted from the lead author's student T. Rausch doctoral thesis. It shows a system of closed curves being geodesic circles. Each of those geodesic circles contains only points that have the same geodesic distance to the center.

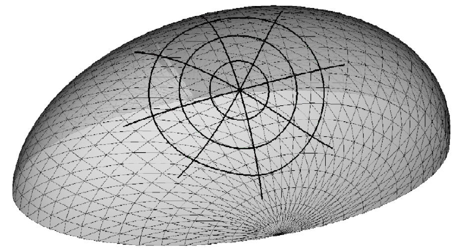

Figure 14. System of Geodesic Circles

All the figures 1-3 stemming from project Riemann code show systems of geodesics on surfaces emanating from a common point. Geodesics are defined as solutions to certain second differential equation systems. All surface curves being shortest joints of their end points are geodesics. Hence computing geodesics is important in order to compute a geodesic circle with center p and given radius ϵ. This geodesic circle contains all the end points of all shortest surface paths from p to any surface point if the path has length ϵ, cf. figur 14 and cf. also figures 1-3.

Geodesics are locally shortest surface curves. This means a sufficiently small subarc of a geodesic is a shortest surface path joining the end points of the subarc. A geodesic (being a locally shortest surface path) may not be the (globally) shortest connection of its endpoints as can be seen by taking a geodesic circular arc on the unit sphere that starts at the southpole and ends shortly after passing through the north pole. Any geodesic will only be the shortest joint of its end points until it intersects with another geodesic joining the same end points. This happens in figure 1 where two geodesics emanating from the same initial point intersect. Both of those intersecting geodesics are not shortest joints between their common start point and any point on the path the geodesic will meet after passing the intersection point.

5.1 Geodesic Medial Curves on Free Form Surfaces

A basic element for the computation of Geodesic Voronoi Diagrams as well as for the computation of the Geodesic Medial Axis will be the computation of medial curves. Those are curves containing only points being equidistantial to two given surface curves. Equidistantial here is understood with respect to the geodesic distance on the surface. The Figure 15 shows such a medial curve that contains only points being geodesically equidistantial with respect to both of the two neighboring curve point sets! The methods for computing such curves are explained with details in [3].

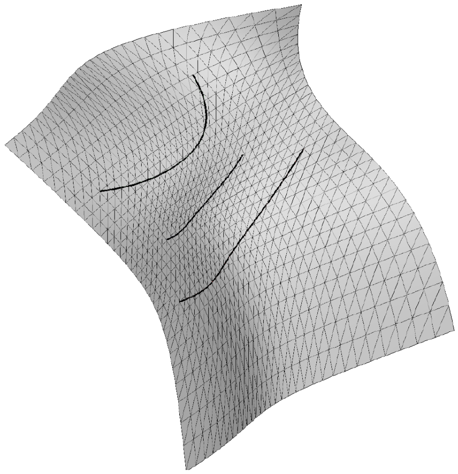

Figure 15. Geodesic Medial Curve

5.2 Geodesic Voronoi Diagrams

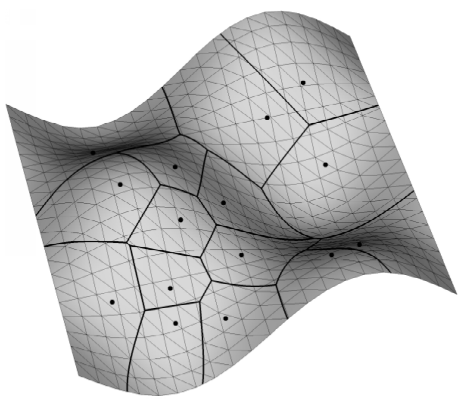

Figure 16. Geodesic Voronoi Diagram on a wave like surface

Figures 16, 17 and 18 show geodesic Voronoi Diagrams on surfaces, cf. [4]. The Geodesic Voronoi Diagram are defined with respect to the finite point set P (given by the black dots) on the surface S. Any of those geodesic Voronoi Diagrams are given by the black edges. Each of those edges consists of points that are (geodesically) equidistant with respect to two (specific) points of the discrete set P. Here the geodesic distance between two given surface points is defined by the minimal length of all surface curves joining the two given points. As in the Euclidean case the geodesic Voronoi Diagram partitions the surface into separate open domains. Each of those open domains contains exactly one point p of P. Together with p, the respective open domain contains all surface points whose geodesic distance to p is

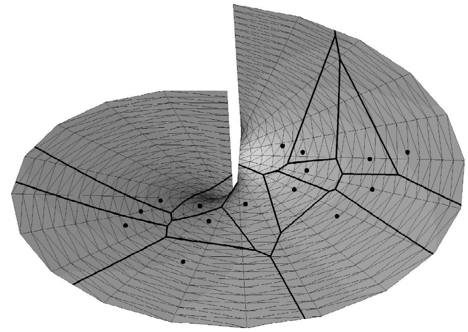

Figure 17. Geodesic Voronoi Diagram on a helicoid

smaller than to any other point in the set P. The edges of the geodesic Voronoi Diagram are built by medial curves containing points that are equidistantial with respect to two given points (or equivalently to two small geodesic circles both of some small radius ϵ). This explains why for this computation we can use the previous experience from computing medial curves.

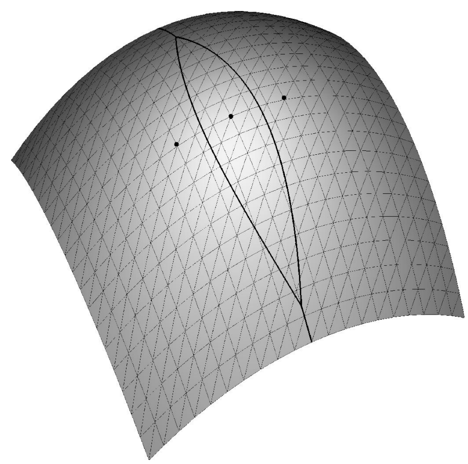

Figure 18. Geodesic Voronoi Diagram with a proximity region bordered by only two edges

5.3 Geodesic Medial Axes

The pictures left and right in figure 19 show geodesic medial axes. They are adapted from the master thesis of the first author's student M. Baer. Those figures illustrate the tree-like structure of the geodesic Medial Axis on curved (simply connected) surface patches, see also figures 20 and 21 that are adapted from M. Baer's master thesis. The shape

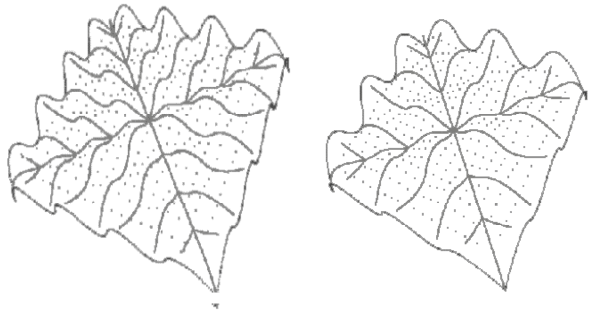

Figure 19. Geodesic Medial Axis

of these patches resemble that of hilly landscapes. The dark curves depict both the boundaries of these surface patches, and the geodesic medial axes that here are topological trees. In analogy to the Euclidean situation every point of the geodesic Medial Axis on a bordered surface B is the center of a geodesic disc of maximal size that is still contained in B. On any surface S a geodesic disc $D(p, r)$ with center p and radius r contains all points on S whose (minimal) geodesic distance to p is smaller or equal to r. The minimal geodesic distance between two points p and q on a surface S is defined by the minimal length of all curves in S that join p and q.

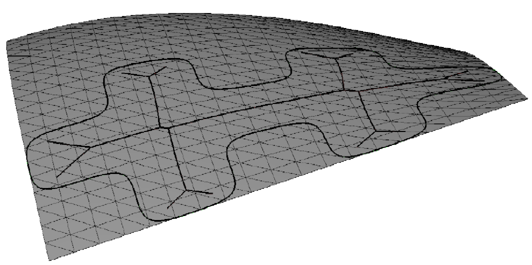

Figure 20. Geodesic Medial Axis of a bordered subsurface of a NURBS surface patch

5.4 Computational Concepts, Accuracy and Speed

The essential parts of Voronoi Diagrams and Medial Axes are built by curve arcs being geodesic medial curves. Numerical tests and computational experiments done in the context of [3] and further studies done later in O. Sniehotta's senior thesis and various test done within the context of R. Kunze's master thesis and later research done at our lab indicate that with our methods it appears to be possible to compute the medial curve with a positional accuracy of 10^{-12} when all the involved computations are done with

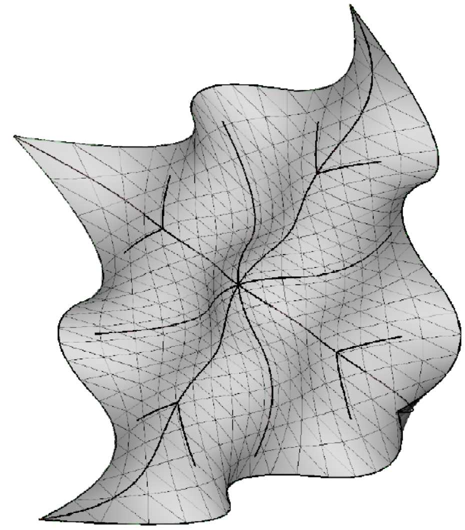

Figure 21. Geodesic Medial Axis of a bordered NURBS surface patch

double precision. Our fastest computing machine used for these computations was an SGI Octane equiped with two MIPS R10000 processors (250 Mhz) - of 1998 meanwhile being considered as fairly slow - would compute medial curves (depending on their length) within less than 15 seconds and would compute a complete Voronoi Diagram (depending on the number of points involved) within a few minutes. Most computations done for the Voronoi Diagrams took place in 1997 and were done on a machine much slower than the Octane purchased for our laboratory in 1998.

The computations for the Geodesic Medial Axes could take several minutes on the Octane depending on the number of edges that the Medial axis tree would have. It should be said that the used algorithms were not completely optimized neither was their implementation. All the algorithms applied in this context used differential equation methods for computing medial curves. In this context it was also necessary to employ differential equation methods as to compute tangent vectors of *geodesic offset* curves.

A geodesic offset on a surface S is defined with respect to some given parametrized progenitor curve $c(t)$ on the surface S. A geodesic offset with geodesic distance s from the progenitor curve is given by a curve $t \to O(s, t)$ where each point $O(s, t)$ is an end point of a geodesic on S that starts at point $c(t)$ with an initial direction orthogonal to the progenitor curves tangent $c'(t)$. The orientation of the aforementioned starting directions of the family of geodesics must be chosen consistently such that the curve $t \to O(s, t)$ (or more general the function $O(s, t)$) will become a continuous function. To show the latter continuity property we also use the fact that the endpoints of the geodesics are obtained by solving an initial value problem. This initial value problem yields the geodesics as solutions of the geodesic differential equation system requesting as input the geodesics starting point and starting vector. Both initial data are obviously continuously dependent on the progenitor curves parameter t. Employing the latter continuity it is a standard consideration to prove that the offset function $(s, t) \to O(s, t)$ will be continuous.

Taking the bisector of two tangent vectors of two appropriately chosen intersecting geodesic offsets will give the medial curve's tangent vector because the medial curve can be viewed as curve (locus) where appropriate systems of geodesic offset curves intersect (see figure 22).

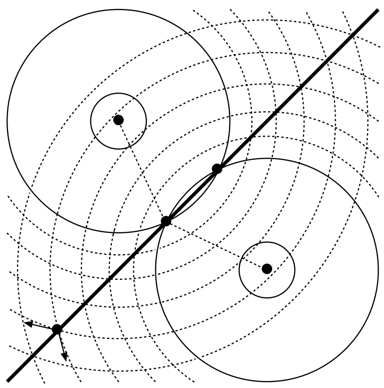

Figure 22. Two systems of offset curves intersecting on a medial curve

The fact that our implementations were not yet optimized combined with the increased speed of modern processors indicate that nowadays as of June 2000 the computation time for Voronoi Diagrams and Medial Axes can still be significantly reduced. This improvement might easily mean a factor of 10 when compared with the computations of 1996 or so when [3] came out.

6 Practical Applications

6.1 Shape Reconstruction and Design using the Medial Axis

The following considerations show that the Medial Axis Transform i. e. the information contained in the Medial Axis and in the *maximal disc radius function* can be used to reconstruct the shape of the solid simply by building the union of the maximal discs as to reconstruct the solid reconstruction theorem cf. [2]. This will also give the solid's boundary surface constructed as an envelope surface with respect to the maximal discs as indicated in the figures 23 and 25 below. It is obvious that keeping the Medial Axis of a solid fixed and modifying the maximal disc radius function slightly will change the solid's shape.

Reconstruction Theorem: Assume D is a solid in R^n with ∂D a closed topological $(n-1)$-dim manifold. If for D the Medial Axis $M(D)$ and the *maximal* **radius function**

$$r : M(D) \to R, \qquad r(x) = d(x, \partial D) \tag{1}$$

are given, then we can reconstruct the solid namely

$$D = \bigcup_{x \in M(D)} K(x, r(x)) \tag{2}$$

with $K(x, r(x))$ beeing a closed disc with center x and radius $r(x)$.

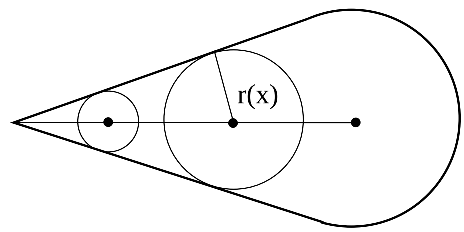

Figure 23. Representing a solid by a union of maximal discs

6.2 Shape Design by Modifying the Radius Function r(x)

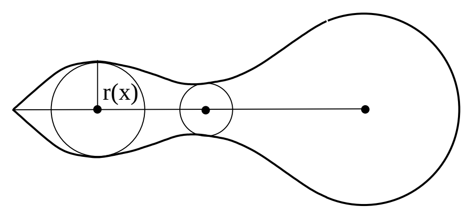

Figure 24. Design (modifications) based on the Medial Axis Transform

Obviously keeping the Medial Axis fixed and modifying the maximal disc radius function r(x) will change the Shape of the solid as indicated in figure 24. Clearly mild uniform growing of the radius function will result in a global fattening of the shape while decreasing the radius function will give a global thinning of the shape. The boundary surface of the solid (corresponding to the given Medial Axis and the given maximal disc radius function) are defined by the envelope curve (envelope surface respectively for 3D Solids) of the family of maximal spheres. This has been known since long time, cf. [1] and see the figure below adapted from the first author's student A. Howinds master thesis showing a family of circles. The envelope curve of those circles is the boundary of the 2D-solid in the plane. The centers of the maximal circles (discs) are located on the Medial Axis of the 2D-solid.

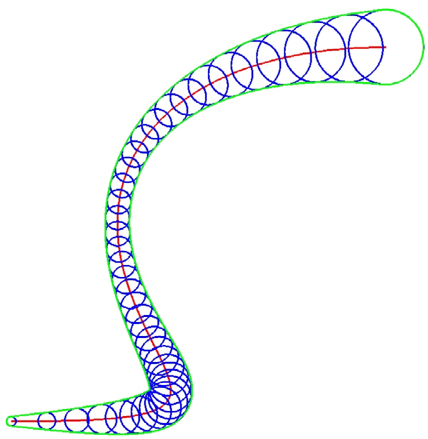

Figure 25. Design (modifications) based on the Medial Axis Transform

Assume that the Medial Axis curve (or surface) is presented locally by a parametrizing function $x(u)$ having a continuous derivative $x'(u)$[1]. Then it is possible to compute very rapidly the contact points of the maximal disc with the envelope surface (being the solid's boundary) if at some Medial Axis point $x(u)$ the derivative (information) $x'(u)$ and the value of the maximal disc radius function $r(u)$ as well as the derivative $r'(u)$ are known. This allows to construct modeling systems that respond and display in real time precisely the complete results of modifications of the maximal disc radius function. *In this context it is possible that a designer indicates a location on the solid's boundary surface where modification (slimming or fattening) is desirable and the system will respond in real time displaying the new solid's boundary locally and globally with the requested modification, cf. figures 24, 25 and 8. An intuitive user interface imitating some features of shape moulding should probably operate in an analog way such that the control over the maximal disc radius function might be exercised by performing pressure i.e. squeezing some kind of mouse ball.* The subsequent figures adapted from A. Howind's master thesis indicate in a prelimary study the results of modeling efforts based on the concepts outlined above. Certainly it would be desirable to incorporate many more (moulding related) features into the system such as removing of the solid's material as to make the solid thinner. This can be done as well employing the precedingly outlined geometric concepts. Somewhat more difficult would be to integrate more complex shape modification features into the system such as changing the shape by attaching new medial arcs (or surface pieces) as new branches to the Medial Axis cf. 7. Doing this would be important as to allow major shape changes from the original solid within the presented geometric concept. One would have to make arrangements that the system automatically blends together nicely the *old parts* of the solid and Medial Axis (where little has been changed) with the new drastically modified parts of the solid and the Medial Axis. The latter modification and blending problems will become even more complicated when the solid is supposed to change drastically the homotopy type say e.g. because one wants to have a hole in the solid implying a hole in the Medial Axis because of the topological shape theorem (see also figure 12).

The following figure 26 from A. Howinds master thesis show a simple prototype modeler offering shape modeling design features employing also radius control of the maximal disc radius function. The various little circles in the figure indicate values of the disc radius function.

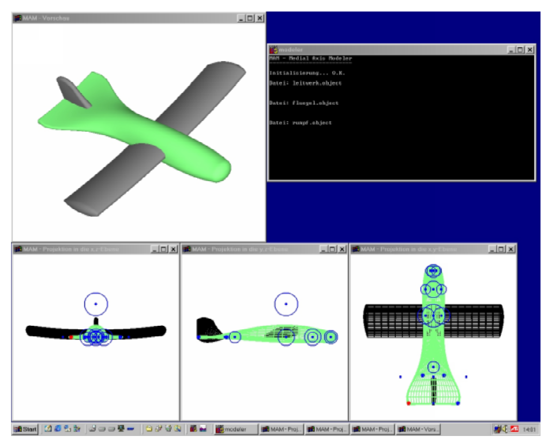

Figure 26. Medial Modeler

The next figure 27 shows design possibilities of the system in detail modifying a solid initially chosen to be flat by starting with a Bezier type medial surface of rectangular shape as the initial Medial Axis control points are chosen to be (uniformly) distributed in a plane. Similiar to the preceding figure the little circles in this figure indicate the values of the disc radius function.

[1] The Medial Axis could be given also by an implicit function then modified methods would work as well.

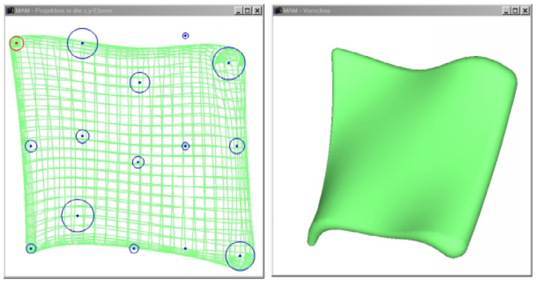

Figure 27. Medial Modeler

Finally the last figure 28 shows two solids where the radius functions have been growing too much such that we have selfintersections of the solids. This must be excluded by geometric criteria providing critical bounds for the radius function.

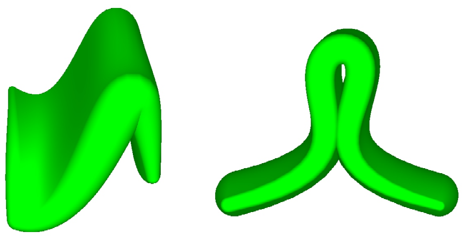

Figure 28. Selfintersecting surfaces supposed to create a solid's boundary

7 Conclusions

In this paper we gave an overview of results treating two fundamental problems in Solid Modeling. The first problem 2.1 was calling for methods and criteria checking approximate shape identity of solids. For this purpose we sketched two methods. A global method 4.1 investigating shape similarity by comparing spectra of the corresponding Laplace Operators that appear to offer a system of interesting new shape invariants based on the fact that isometric surfaces and solids must have the same spectra. The other method 4.2 employed the curvature line wire frame surface model and its singularities (umbilics) to assess approximate shape identity for solids. If a closed surface has only stable umbilics then a sufficiently small deformation of the surface will give a surface with a curvature line wire frame and stable umbilics that can be approximately matched with the corresponding curvature line wire frame model and with the umbilics of the old surface. If the umbilics are finite and stable then there will be a one to one correspondence between the umbilics on both surfaces. This correspondence will also respect the types of the umbilics. Most material in this paper is related to the Medial Axis Transform. This material covers the topological shape theorem for solids stating that the Medial Axis is a deformation retract of its reference solid if the solid's boundary is curvature continuous. (This regularity assumption for the boundary can be relaxed further.) It is explained that the concept of Medial Axis and Voronoi diagrams can be extended naturally to free form surfaces where these objects Medial Axis and Voronoi Diagrams are now defined and computed using the geodesic distance on the surface. The latter defines the distance between two surface points by the length of the shortest surface path joining these points. Finally this paper states a theorem saying that (even under very weak regularity assumptions for the solid's boundary) it is always possible to reconstruct the solid from a union of the maximal discs having their centers on the Medial Axis. Hence knowing the Medial Axis and the associated maximal disc radius function it is easy to reconstruct the solid. It is explained that this result offers also a new concept to construct a user interface allowing to modify and design a solid's shape by changing its maximal disc radius function. *The design modification results in fattening and thinning of the shape in selected regions of the solids, if the radius function is increased or decreased respectively in the related regions of the Medial Axis. Controlling this shape modification could probably be done most intuitively via a haptic device allowing to shrink the maximal disc radius by squeezing e.g. an elastic ball mouse after positioning the ball (semi automatically) with its center in an appropriately chosen (influence) area of the Medial Axis. This approach offers new possibilities for intuitive shape moulding devices appealing to the human haptic interaction with the world in ways that appear to be closer to moulding than currently used devices.*

8 Acknowledgements

Even in modern times with all kinds of electronic support preparing (merely) a short scientific paper is still an extremely cumbersome and time consuming enterprise. A person who has not been involved in doing this work usually completely underestimates the necessary time by factors. (Sometimes even people who have experience underestimate the effort that will be needed to do a reasonable job.) This paper could not have been prepared timely enough to meet the submission deadline and including all the material and figures etc. if the following persons would not have helped: G. Böttcher, G. Demirci, A. Kaiser, N. Peinecke, M. Reuter. The authors thank them kindly for their generous support. The authors also thank Ms. T. Rayle who helped improving the english language.

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